UDC 62-50

NONLINEAR PROBLEM OF EVASION OF CONTACT WITH A TERMINAL SET OF COMPLEX STRUCTURE

PMM Vol. 39, №1, 1975, pp. 3-11

A. A. CHIKRII
(Kiev)

(Received December 28, 1973)

We examine the problem posed in [1, 2], of the evasion of a conflict-controlled motion from a given set. We investigate the case of a nonlinear system of differential equations which specify the dynamics, and of a terminal set of complex structure. We have obtained sufficient conditions for evasion. As an application we examine the problem of evasion in differential games with phase constraints and the problem of escaping from many pursuers. We illustrate the results obtained by examples.

1. Let the law of motion of an object be given by the equation

$$z^{\cdot} = f(z, u, v), \quad z \in E^n \tag{1.1}$$

Here the control parameters u and v are chosen from sets U and V belonging to E^n. A terminal set M is specified. The game is played by two players P and E, who influence system (1.1) by means of controls u and v. Player P tries to lead out the trajectory of (1.1) onto set M, while player E hinders this action.

The game parameters satisfy the following requirements:

(1). The function $f(z, u, v)$ is continuous in the arguments and is continuously differentiable in z.

(2). The sets U and V are compact.

(3). The set $f(z, U, v)$ is convex for any z and v, $v \in V$.

(4). A constant C exists such that

$$|(z, f(z, u, v))| \leqslant C(1 + \|z\|^2)$$

(5). The set M is specified as follows:

$$M = \bigcup_{i=1}^{r+q} M_i \tag{1.2}$$

$$M_i = \left\{ z: \begin{array}{l} \varphi_{ij}(z) = 0, \ j = 1, 2, \ldots, m(i) \\ \varphi_{ij}(z) \leqslant 0, \ j = m(i)+1, \ m(i)+2, \ldots l(i) \end{array} \right\}, \quad i = 1, 2, \ldots, r$$

$$M_i = \{z: \varphi_i(z, p) \leqslant 0, \ \forall p \in E^n\}, \quad i = r+1, \ r+2, \ldots, r+q$$

Here $\varphi_{ij}(z)$ are continuously differentiable functions, $\varphi_i(z, p) = (z, p) - W_{M_i}(p)$, $W_{M_i}(p)$ is the support function of the closed convex set M_i.

Let us recall the definition of ε-strategies [3] which we shall use subsequently.

Definition 1. We say that an ε-strategy (Γ_E) of player E is given if for each point $z \in E^n$ there have been determined a number $\varepsilon(z)$, $\varepsilon(z) > 0$, and a function $\Gamma_E(t; z)$, $0 \leqslant t \leqslant \varepsilon(z)$, satisfying the following condition: $v(t) = \Gamma_E(t; z)$ is a measurable function of t, taking values in set V.

Definition 2. We say that an ε-strategy (Γ_P) of player P is specified if for each point $z \in E^n$ there has been determined a function $\Gamma_P(t; \varepsilon, v(\cdot), z)$ which associates with point z, with a number $\varepsilon > 0$, and with the function $v(t), 0 \leqslant t \leqslant \varepsilon$, a function $u(t) = \Gamma_P(t; \varepsilon, v(\cdot), z)$, measurable for $0 \leqslant t \leqslant \varepsilon$, taking values in U.

Definition 3. We say that a trajectory $z(t)$, starting at a point z_0, has been determined on the half-open interval $[0, t_0)$ or on the closed interval $[0, t_0]$ if $z(t)$ is an absolutely continuous function of t, $z(0) = z_0$, and a set $T \subset [0, t_0)$ (respectively, $T \subset [0, t_0]$) exists such that

a) $0 \in T$ and if $\tau \in T$ and $\varepsilon(z(\tau)) + \tau < t_0$ ($\tau + \varepsilon(z(\tau)) \leqslant t_0$ for $[0, t_0]$), then $\tau + \varepsilon(z(\tau)) \in T$ and the interval $(\tau, \tau + \varepsilon(z(\tau)))$ does not contain points of T;

b) the set $T \cup \{t_0\}$ closed;

c) if we denote $\tau_0 = \sup\{\tau: \tau \in T\}$, then the function $z(t)$ satisfies almost everywhere the equation $z^{\cdot} = f(z, u(t), v(t))$, $v(t) = \Gamma_E(t - \tau; z(\tau))$, $u(t) = \Gamma_P(t - \tau; \varepsilon(z(\tau)), v(\cdot), z(\tau))$ on each interval $[\tau, \tau']$, $\tau' = \tau + \varepsilon(z(\tau)) < t_0$, or on the interval $[\tau_0, t_0)$ ($[\tau_0, t_0]$);

d) in the case of the closed interval $[0, t_0]$, if $\tau_0 = t_0$, then $t_0 \in T$.

The trajectory $z(t) \equiv z(t; z_0, \Gamma_P, \Gamma_E)$ is uniquely defined on the whole semi-interval $[0, \infty)$ by giving an initial point z_0 and the strategies Γ_P and Γ_E (see [3, 4]).

We say that an evasion from contact with set M from a point z_0 is possible in game (1.1), (1.2) if a strategy Γ_E exists such that for any strategy Γ_P the trajectory $z(t) \equiv z(t; z_0, \Gamma_P, \Gamma_E)$ does not hit onto the set M for $0 \leqslant t < \infty$.

2. We shall subsequently use the following notation:

$$P_i(z) = \{p: \|p\| = 1, \ \varphi_i(z, p) \geqslant 0\}, \quad i = r+1, r+2, \dots, r+q$$

The time derivatives of the functions $\varphi_i(z, p)$, $\varphi_{ij}(z)$, by virtue of system (1.1) for fixed p, u, v are

$$\varphi_i^{(k)}(z, p) \equiv \frac{d^k}{dt^k}\varphi_i(z, p) = (\nabla\varphi_i^{(k-1)}(z, p), f(z, u, v))$$

$$\nabla\varphi_i^{(0)}(z, p) = p, \quad i = r+1, r+2, \dots, r+q, \quad k = 1, 2, \dots$$

$$\varphi_{ij}^{(k)}(z) \equiv \frac{d^k}{dt^k}\varphi_{ij}(z) = (\nabla\varphi_{ij}^{(k-1)}(z), \ f(z, u, v))$$

$$\nabla\varphi_{ij}^{(0)}(z) = \nabla\varphi_{ij}(z), \quad i = 1, 2, \dots, r, \quad j = 1, 2, \dots, l(i), \quad k = 1, 2, \dots$$

Let S_i be an open set containing M_i

$$S_i^* = S_i \setminus M_i, \quad S^* = \bigcup_{i=1}^{r+q} S_i^*, \quad S = S^* \setminus M \tag{2.1}$$

$$I = \{1, 2, \dots, r+q\}, \quad I_1 = \{1, 2, \dots, r\}, \quad I_2 = \{r+1, r+2, \dots, r+q\}$$

For $z \in S$ we set

$$I(z) = \{i: i \in I, z \in S_i^*\}, \ I_1(z) = I(z) \cap I_1, I_2(z) = I(z) \cap I_2$$

3. We state conditions each succeeding one of which assumes the fulfillment of the preceding ones.

Condition 1. The function $f(z, u, v)$ is differentiable in z up to order k_*-1, while each of the functions $\varphi_{ij}(z)$ is continuously differentiable up to order k_* inclusive.

Condition 2. If $z \in S$ and $i \in I_2(z)$, then there exist a vector $p \in P_i(z)$ and number $k_i = k_i(z) \leqslant k_*$ such that the functions $\varphi_i^{(\nu)}(z, p)$ $(\nu = 1, 2, \ldots, k_i - 1)$ do not depend upon u and v; moreover, $\varphi_i^{(\nu)}(z, p) \geqslant 0$ $(\nu = 1, 2, \ldots, k_i - 1)$, while

$$\varphi_i^{(k_i)}(z, p) = (\nabla\varphi_i^{(k_i-1)}(z, p), f(z, u, v))$$

In the case $z \in S$, $i \in I_1(z)$, either (a) numbers $\gamma = \gamma(i)$ $(1 \leqslant \gamma(i) \leqslant l(i))$ and $k_i^1 = k_i^1(z, \gamma) \leqslant k_*$ exist such that $\varphi_{i\gamma}(z) > 0$, the functions $\varphi_{i\gamma}^{(\nu)}(z)$ $(\nu = 1, 2, \ldots, k_i^1 - 1)$ do not depend upon u and v; moreover, $\varphi_{i\gamma}^{(\nu)}(z) \geqslant 0$ $(\nu = 1, 2, \ldots, k_i^1 - 1)$, while

$$\varphi_{ij}^{k_i^1}(z) = (\nabla\varphi_{i\gamma}^{(k_i^1-1)}(z), f(z, u, v))$$

or (b) $\varphi_{ij}(z) < 0$ for all j, $1 \leqslant j \leqslant l(i)$, but numbers $\mu = \mu(i)$ $(1 \leqslant \mu(i) \leqslant m(i))$ and $k_i^2 = k_i^2(z, \mu) \leqslant k_*$ exist such that the functions $\varphi_{i\mu}^{(\nu)}(z)$ $(\nu = 1, 2, \ldots, k_i^2 - 1)$ do not depend upon u and v; moreover, $\varphi_{i\mu}^{(\nu)}(z) \leqslant 0$ $(\nu = 1, 2, \ldots, k_i^2 - 1)$, while

$$\varphi_{i\mu}^{k_i^2}(z) = (\nabla\varphi_{i\mu}^{(k_i^2-1)}(z), f(z, u, v))$$

In what follows cases (a) and (b) of Condition 2 are designated 2a and 2b, respectively.

Condition 3. The system of inequalities

$$\min_{u \in U}(\nabla\varphi_i^{(k_i-1)}(z, p), f(z, u, v)) \geqslant \sigma(z), \quad i \in I_2(z) \tag{3.1}$$

$$\min_{u \in U}(\nabla\varphi_{i\gamma}^{(k_i^1-1)}(z), f(z, u, v)) \geqslant \sigma(z), \quad i \in I_1(z) \ (2a)$$

$$\max_{u \in U}(\nabla\varphi_{i\mu}^{(k_i^2-1)}(z), f(z, u, v)) \leqslant -\sigma(z), \quad i \in I_1(z) \ (2b)$$

where $\sigma(z)$ is some continuous function, strictly positive in any bounded region, is solvable relative to v, $v \in V$, at each point $z_0 \in S$.

Suppose that conditions 1 – 3 have been satisfied. For the point $z \in S$ and for $i \in I(z_0)$ we fix $p^\circ \in P_i(z_0)$, k_i, k_i^1, k_i^2, $\gamma(i)$, $\mu(i)$ and $v_0 = v(z_0)$, satisfying (3.1). Let us consider the functions

$$\chi_i^{z_0}(z) = \min_{u \in U}(\nabla\varphi_{i\gamma}^{(k_i^1-1)}(z), f(z, u, v_0)), \quad i \in I_1(z_0) \ (2a)$$

$$\varkappa_i^{z_0}(z) = \max_{u \in U}(\nabla\varphi_{i\mu}^{(k_i^2-1)}(z), f(z, u, v_0)), \quad i \in I_1(z_0) \ (2b)$$

$$\psi_i^{z_0}(z) = \min_{u \in U}(\nabla\varphi_i^{(k_i-1)}(z, p^\circ), f(z, u, v_0)), \quad i \in I_2(z_0)$$

If z_0 is replaced by some set Z, $Z \subset S$, while for each z_0, i takes values from $I(z_0)$, then we obtain a family of continuous functions which we denote

$$\{\lambda_i^{z_0}(z)\}_{z_0 \in Z,\, i \in I(z_0)} \tag{3.2}$$

Condition 4. The family of functions (3.2), where Z is a bounded set, is equicontinuous on set Z.

4. Theorem on evasion of contact. In the differential game (1.1), (1.2) let there exist a number k_* and a set S such that Conditions 1 – 4 are satisfied. Then evasion from contact with the terminal set is possible for any point z_0, $z_0 \overline{\in} M$.

Proof. Let $z_0 \in S$. For $i \in I_1(z_0)$, by virtue of Conditions 2, 3 either numbers $\gamma = \gamma(i)$, k_i^1 and $v_0 = v(z_0) \in V$ exist such that

$$\min_{u\in U} (\nabla\varphi_{i\gamma}^{(k_i^1-1)}(z_0), f(z_0, u, v_0)) \geqslant \sigma(z_0) > 0$$

or numbers $\mu = \mu(i)$, k_i^2 and $v_0 = v(z_0) \in V$ exist such that

$$\max_{u\in U} (\nabla\varphi_{i\mu}^{(k_i^2-1)}(z_0), f(z_0, u, v_0)) \leqslant -\sigma(z_0) < 0$$

In the first case we select a neighborhood Ω_{r_i} $(r_i > 0)$ of point z_0 so small that the inequality

$$\min_{u\in U} (\nabla\varphi_{i\gamma}^{(k_i^1-1)}(z), f(z, u, v_0)) \geqslant 0 \tag{4.1}$$

is satisfied by continuity, while in the second case, the inequality

$$\max_{u\in U} (\nabla\varphi_{i\mu}^{(k_i^2-1)}(z) f(z, u, v_0)) \leqslant 0 \tag{4.2}$$

and

$$\Omega_{r_i} \cap (\bigcup_{i\overline{\in}I(z_0)} M_i) = \varnothing$$

For $i \in I_2(z_0)$, by virtue of Conditions 2, 3 a vector $p \in P_i(z_0)$ number k_i and $v_0 = v(z_0) \in V$ exist such that $\min\limits_{u\in U} (\nabla\varphi_i^{(k_i-1)}(z_0, p), f(z_0, u, v_0)) \geqslant \sigma(z_0) > 0$. We select a neighborhood Ω_{r_i} of point z_0 such that the inequality

$$\min_{u\in U} (\nabla\varphi_i^{(k_i-1)}(z, p), f(z, u, v_0)) \geqslant 0 \tag{4.3}$$

is satisfied by continuity and

$$\Omega_{r_i} \cap (\bigcup_{i\overline{\in}I(z_0)} M_i) = \varnothing$$

We set

$$r_0 = \min_{i\in I(z_0)} r_i$$

From the assumptions on sets U and V and on function $f(z, u, v)$ and from the Gronwall lemma [5] follows the existence of $\tau_0 > 0$ such that a trajectory starting at point z_0 with an arbitrary measurable control $u(t)$ and with $v(t) = v_0$ does not leave the neighborhood Ω_{r_0} during time τ_0. Let us construct the evasion strategy Γ_E^*. To do this we set $\varepsilon(z_0) = \tau_0$ and $v(t) = v_0$, $0 \leqslant t \leqslant \tau_0$. Then, the control $u(t)$ is determined in accord with strategy Γ_P and system (1.1) can be integrated on the interval $[0, \tau_0]$, obtaining trajectory $z(t)$.

Let $z_0 \overline{\in} S$. We select the neighborhood Ω_{r_0} of point z_0 such that $M \cap \Omega_{r_0} = \varnothing$. Then $\tau_0 > 0$ exists such that the trajectory starting at point z_0 with arbitrary measurable controls $u(t)$ and $v(t)$ does not leave the neighborhood Ω_{r_0} during time τ_0. We set $\varepsilon(z_0) = \tau_0$ and, having chosen a measurable $v(t)$, $0 \leqslant t \leqslant \tau_0$, with values in V, we determine the strategy Γ_E^*. Then the control $u(t)$ is determined in accordance with strategy Γ_P and system (1.1) can be integrated on the interval $[0, \tau_0]$, obtaining trajectory $z(t)$.

Let us show that a trajectory of system (1.1), not intersecting set M at a finite instant, corresponds to the strategy pair (Γ_P, Γ_E^*) and to the point z_0 ($z_0 \overline{\in} M$) To do this we establish estimates for the measurement of functions $\varphi_{ij}(z)$ and $\varphi_i(z, p)$ along trajectory $z(t)$, corresponding to the strategy pair (Γ_P, Γ_E^*). For $z_0 \in S$ and $i \in I(z_0)$, according to Taylor's formula with a definite integral as the remainder term [6], the functions $\varphi_{i\gamma}(z)$ from 2a, $i \in I_1(z_0)$, $\varphi_{i\mu}(z)$ from 2b, $i \in I_1(z_0)$, and $\varphi_i(z, p)$, $i \in I_2(z_0)$ can be represented along the trajectory $z(t)$, $0 \leqslant t \leqslant \tau_0$, in the form

$$\varphi_{i\gamma}(z(t)) = \sum_{j=0}^{k_i^1-1} \frac{t^j}{j!} \varphi_{i\gamma}^{(j)}(z_0) + \int_0^t \frac{(t-\tau)^{k_i^1-1}}{(k_i^1-1)!} (\nabla \varphi_{i\gamma}^{(k_i^1-1)}(z(\tau)), f(z(\tau), u(\tau), v_0))\, d\tau \qquad (4.4)$$

(the other two representations are analogous to (4.4)).

By the definition of strategy Γ_E^*, $z(\tau) \in \Omega_r$, for $\tau \leqslant \tau_0$. Using inequalities (4.1)–(4.3) and Condition 2, from the representations of type (4.4) we obtain

$$\begin{aligned}
&\varphi_{i\gamma}(z(t)) \geqslant \varphi_{i\gamma}(z_0) > 0, \quad 0 \leqslant t \leqslant \tau_0, \quad i \in I_1(z_0) \ \ (2a) \\
&\varphi_{i\mu}(z(t)) \leqslant \varphi_{i\mu}(z_0) < 0, \quad 0 \leqslant t \leqslant \tau_0, \quad i \in I_1(z_0) \ \ (2b) \\
&\varphi_i(z(t), p) > \varphi_i(z_0, p) \geqslant 0, \quad 0 < t \leqslant \tau_0, \quad i \in I_2(z_0)
\end{aligned} \qquad (4.5)$$

Thus, for $z_0 \in S$ and for each $i \in I(z_0)$ the functions $\varphi_{i\gamma}(z)$, $\varphi_i(z, p)$ grow monotonically, while the functions $\varphi_{i\mu}(z)$ decrease monotonically along trajectory $z(t)$ during some time. Hence it follows that trajectory $z(t)$ does not intersect the set $\bigcup_{i \in I(z_0)} M_i$ on the time interval $[0, \tau_0]$. But since, by the construction of strategy Γ_E^*, $z(t)$ does not intersect $\bigcup_{i \in I(z_0)} M_i$ either during time τ_0, we have that $z(t)$ does not intersect M on the interval $[0, \tau_0]$.

For $z_0 \overline{\in} S$, it also follows from the definition of strategy Γ_E^* that $z(t)$ does not intersect M on some interval $[0, \tau_0]$. Consequently, if $\tau, \tau' \in T$, $\tau' = \tau + \varepsilon(z(\tau))$, then trajectory $z(t)$ does not intersect M on the interval $[\tau, \tau']$.

Let us show that in any bounded subset Z of set S we can choose $\varepsilon(z) \geqslant \tau > 0$, where the constant τ depends only upon set Z. We denote $\bar{Z}$ as the closure of set Z, $\min_{z \in \bar{Z}} \sigma(z) = \Delta$. According to Condition 4 the family of functions (3.2) is equicontinuous on Z, i.e. for $\Delta > 0$ there exists $\delta_1 > 0$ such that

$$|\lambda_i^{z_0}(z_1) - \lambda_i^{z_0}(z_2)| \leqslant \Delta$$

for all z_1, z_2 from Z such that $\|z_1 - z_2\| \leqslant \delta_1$ and for all $z_0 \in Z$ and $i \in I(z_0)$. Thus, for any point $z_0 \in Z$ each of the functions $\lambda_i^{z_0}(z)$, $i \in I(z_0)$, is nonnegative or nonpositive in a neighborhood of radius not less than δ_1 of this point. In addition, there exists $\delta_2 > 0$, such that for any point $z_0 \in Z$ its neighborhood Ω_{δ_2} does not intersect the set $\bigcup_{i \in I(z_0)} M_i$. We set $\delta = \min(\delta_1, \delta_2)$. Since trajectory $z(t)$ satisfies in Z a Lipschitz condition with constant L, for any point $z \in Z$ we can choose.

$$\varepsilon(z) \geqslant \delta/L = \tau > 0$$

Let us now assume that a trajectory $z(t)$, starting from a point $z_0 \in M$ and corresponding to the strategy pair (Γ_P, Γ_E^*), first intersects the boundary of M at some finite instant t_* i.e. for some i either $\varphi_{ij}(z(t_*)) = 0$ for any $j = 1, 2, \ldots, m(i)$ and $\varphi_{ij}(z(t_*)) \leqslant 0$ for any $j = m(i)+1, m(i)+2, \ldots, m(i)+l(i)$, or $\varphi_i(z(t_*), p) \leqslant 0$ for any $p \in E^n$. But then $\vartheta > 0$ exists such that $z(t)$ belongs to some bounded subset Z of set S for all t, $t_* - \vartheta \leqslant t < t_*$; moreover, by virtue of what has been said, the point t_* must be the limit point for the points of T, corresponding to the trajectory. Since we can select $\varepsilon(z) \geqslant \tau$ in set Z, there exists an instant $t_1 \in T$, $t_* - \vartheta \leqslant t_1 < t_*$, such that $t_* = t_1 + \beta$, where $\beta \leqslant \varepsilon(z(t_1))$. Since $z(t_1) \in S$, by virtue of (4.5) and by continuity one of the relations

$$\varphi_{i\gamma}(z(t)) \geqslant \varphi_{i\gamma}(z(t_1)) > 0, \quad t_1 \leqslant t \leqslant t_1 + \varepsilon(z(t_1))$$

$$\varphi_{i\mu}(z(t)) \leqslant \varphi_{i\mu}(z(t_1)) < 0, \quad t_1 \leqslant t \leqslant t_1 + \varepsilon(z(t_1))$$

$$\varphi_i(z(t), p) > \varphi_i(z(t_1), p) \geqslant 0, \quad t_1 < t \leqslant t_1 + \varepsilon(z(t_1))$$

is satisfied for each $i \in I(z(t_1))$. By the definition of strategy Γ_E^* the trajectory $z(t)$ does not intersect the set $\bigcup_{i \in I(z(t_1))} M_i$, on the interval $[t_1, t_1 + \varepsilon(z(t_1))]$ therefore, we have arrived at a contradiction. The theorem is proved.

5. Let us dwell on a linear system (1.1), $I_1 = \phi$, $q = 1$, which includes the cases treated in [1, 2], and compare the results by examples.

Example. In a Euclidean space E^n, $n \geqslant 2$, the motions of two points x and y, where x is the pursuer, y is the pursued, are given by the equations

$$x^{(r)} + a_1 x^{(r-1)} + \ldots + a_{r-1}\dot{x} + a_r x = u$$
$$y^{(s)} + b_1 y^{s-1} + \ldots + b_{s-1}\dot{y} + b_s y = v$$
$$M = \{(x, y)\colon x = y\}, \; s \leqslant n-1, \; u \in U \in E^n, \; v \in V \in E^n, \; \dim V = n$$

Here $x^{(i)}$, $y^{(i)}$ are derivatives of order i, a_i, b_j are linear mappings of space E^n into itself, U and V are convex compact sets. If one of the following conditions is satisfied: (1) $s < r$, (2) for $s = r$ a vector ω exists such that $W_{V+\omega}(p) - W_U(p) > 0 \quad \forall p \in E^n$, then escape is possible. These conditions are the same as the conditions in [2] for $n \geqslant s+1$. In Pontriagin's check example $(n \geqslant 3)$ and in the "boy and crocodile"(*) problem $(n \geqslant 2)$, being special cases of the example considered, the sufficient conditions for escape agree with the conditions in [1, 7].

6. As an application let us consider the evasion problem under phase constraints [8-13]. Let the state vector z of system (1.1) be constrained by the following restriction: it must not leave a set G, the terminal set M is convex and closed

$$G = \{z\colon \varphi_i(z) < 0, \; i = 1, 2, \ldots, r\} \tag{6.1}$$

$$M = \{z\colon (z, p) \leqslant W_M(p) \;\; \forall p \in E^n\} \tag{6.2}$$

---

*) Editor's Note. The names of games mentioned in this paper in inverted commas are translated verbatim from the Russian original text.

Here $\varphi_i(z)$ are continuously differentiable functions. Player E tries to prevent the contact of a trajectory of system (1.1) with M, without violating the phase constraints (6.1); the aim of player P is to obstruct his opponent. We assume that $M \cap G \neq \emptyset$.

Having set

$$M_i = \{z: -\varphi_i(z) \leqslant 0\}, \quad i = 1, 2, \ldots, r \tag{6.3}$$

$$M = M_{r+1}, \quad M_0^{\cdot} = \bigcup_{i=1}^{r+1} M_i$$

we get that the problem of making system (1.1) evade set M under the constraints (6.1) is reduced to the problem of making system (1.1) evade a set M_0 with no constraints. The latter problem is a special case of the evasion problem in game (1.1), (1.2) with $q = 1, m(i) = 0, l(i) = 1, i = 1, 2, \ldots, r$.

We assume that the set G is closed

$$G = \{z: \varphi_i(z) \leqslant 0, \ i = 1, 2, \ldots, r\} \tag{6.4}$$

and that the inequalities in (6.4) satisfy the Slayter condition. If instead of (6.3) we assume

$$M_i = \{z: -\varphi_i(z) < 0\}, \quad i = 1, 2, \ldots, r$$

we get that the problem of system (1.1) evading M under constraints (6.4) is reduced to the problem of system (1.1) evading a set M_0 without constraints, which set is not closed. In this case, instead of (2.1) we should set

$$S_i^* = \{z: \varphi_i(z) = 0\}, \ i = 1, 2, \ldots, r; \quad S = (S_* \cup \partial G) \setminus M_0$$

where S_* is an open set containing M_0. The theorem on evasion of contact for this case can be proved under Conditions 1–4 without essential changes.

The proposed approach permits us to obtain sufficient evasion conditions in problems of the type "games with a death line", "a cornered rat", "corridor patrolling" [14].

Example. The laws of motion of a pursuing and a pursued objects are given by the equations

$$x^{\cdot\cdot} = u, \quad y^{\cdot} = v, \quad \|u\| \leqslant 1, \quad \|v\| \leqslant 1, \quad M = \{(x, y): \ x = y\}$$

where x, y are vectors in a Euclidean space of dimension $n \geqslant 2$; moreover, the pursued object is constrained by the restriction: $(a, y) > 0$, a is a constant vector. Here $(a, y) = 0$ is the "hyperplane of death". By carrying out the appropriate calculations we get that evasion is possible from any initial position (x, y) such that $x \neq y$, $(a, y) > 0$ and $k_* = 1$.

7. Let us consider the problem of escaping from several pursuers. Let the motion of each of N pursuers be described by the system of Eqs. (7.1), while the pursued moves in accordance with system (7.2)

$$x_i^{\cdot} = f_i(x_i, u_i), \quad i = 1, 2, \ldots, N; \ x_i \in E^{r_i}, \quad u_i \in U_i \subset E^{r_i} \tag{7.1}$$

$$y^{\cdot} = g(y, v), \quad y \in E^{r_0}, \quad v \in V \subset E^{r_0} \tag{7.2}$$

Here u_i, v are the players' controls. Having denoted $z = (x_1, x_2, \ldots, x_N, y)$, on the direct product

$$E^{r_1} \times E^{r_2} \times \ldots \times E^{r_N} \times E^{r_0}$$

we delineate the terminal set M

$$M = \bigcup_{i=1}^{N} M_i, \quad M_i = \{z : \{x_i = y\}_1^e\} \tag{7.3}$$

where $\{x_i\}_1^e = (x_{i1}, x_{i2}, \ldots, x_{ie})$, $\{y\}_1^e = (y_1, \ldots, y_e)$, $e \leqslant \min_{0 \leqslant i \leqslant N} r_i$ are the first e coordinates of the corresponding vectors.

We assume that the conditions analogous to Conditions 1–4 are fulfilled for the system (7.1), (7.2), ensuring the existence, uniqueness and continuability of the solution of (7.1), (7.2). The support function of set M_i has the form $W_{M_i}(p) = 0$ if $\{p_i = -p_0\}_1^e$, $\{p_i\}_{e+1}^{r_i} = 0$, $\{p_0\}_{e+1}^{r_0} = 0$, $p_j = 0\ (j \neq i)$, $W_{M_i}(p) = \infty$ if even one of the indicated conditions is not satisfied. Here $p = (p_1, p_2, \ldots, p_N, p_0) \in E^{r_1} \times E^{r_2} \times \ldots \times E^{r_N} \times E^{r_0}$. We see that the problem of system (7.1), (7.2) evading a set M of form (7.3) is included in scheme proposed earlier.

Example. The laws of motion of the pursuers and of the escaper are given by the equations $\ddot{x}_1 = u_1 \quad \ddot{x}_2 = u_2, \quad \dot{y} = v \quad \| u_1 \| \leqslant \alpha, \quad \| u_2 \| \leqslant \beta, \quad \| v \| \leqslant 1, \quad \alpha, \beta > 1$

$$M = M_1 \cup M_2, \quad M_1 = \{(x_1, y) : x_1 = y\}, \quad M_2 = \{(x_2, y) : x_2 = y\}$$

where x_1, x_2, y are vectors in a Euclidean space of dimension $n \geqslant 2$. It can be verified that evasion is possible from any position such that $x_1 \neq y$, $x_2 \neq y$, with $k_* = 1$.

The author thanks B.N.Pshenichnyi for discussing the results.

REFERENCES

1. Pontriagin, L.S. and Mishchenko, E.F., The problem of evasion of contact in linear differential games. Differentsial'nye Uravneniia, Vol. 7, № 3, 1971.
2. Pontriagin, L.S., A linear differential game of escape. Trudy Matem. Inst. Akad. Nauk SSSR im. Steklova, Vol. 112, 1971.
3. Pchenitchny (Pshenichnyi), B.N., ε-strategies in differential games. In: Topics in Differential Games. New York–London–Amsterdam, 1973.
4. Filippov, A.F., On certain aspects of optimal control theory. Vestn. Moskovsk. Gos. Univ., Ser. Matem., Mekhan., Astron., Fiz., Khim., № 2, 1959.
5. Sansone, J., Ordinary Differential Equations. Vol. 1, Moscow, Izd. Inostr. Lit., 1953.
6. Kartan, A., Differential Calculus. Moscow, "Mir", 1971.
7. Pshenichnyi, B.N. and Chikrii, A.A., Problem of evasion of contact in differential games. (English translation), Pergamon Press, Zh. Vych. Mat., Mat. Fiz., Vol. 14, № 6, 1974.
8. Krasovskii, N.N., Game Problems on the Contact of Motions. Moscow, "Nauka", 1970.
9. Krasovskii, N.N., Encounter-evasion differential game, I, II. Izv. Akad. Nauk SSSR, Tekhn. Kibernetika, №№ 2, 3, 1973.
10. Pshenichnyi, B.N. and Dziubenko, G.Ts., On differential games with two terminal sets. In: Mathematical Methods of Investigating and Optimizing Systems, № 5, Kiev, 1970.
11. Kurzhanskii, A.B., Differential games of encounter under bounded phase coordinates. Dokl. Akad. Nauk SSSR, Vol. 192, № 3, 1970.

12. Nikol'skii, M.S., Differential Games with phase constraints. In : Theory of Optimal Solutions, №4, Kiev, 1969.
13. Friedman, A., Differential games with restricted phase coordinates. J. Diff. Eqts., Vol. 8, №1, 1970.
14. Isaacs, R.P., Differential Games. New York, Johm Wiley and Sons, Inc., 1965.

Translated by N.H.C.

UDC 62-50

ESCAPE OF NONLINEAR OBJECTS OF DIFFERENT TYPES WITH INTEGRAL CONSTRAINTS ON THE CONTROL

PMM Vol. 39, №1, 1975, pp. 12-23
P. B. GUSIATNIKOV
(Moscow)
(Received December 24, 1973)

We quote sufficient conditions for evasion of contact in a game of two nonlinear objects with integral constraints on the control.

1. Let t_0 be a fixed real number. Let the laws of motion of the pursuing vector $x \in E^n$ and of the escaping vector $y \in E^n$ be described for $t \geqslant t_0$ by the vector differential equations

$$\begin{aligned} d^k x / dt^k &= L(t, X) + u, \quad x = \operatorname{col}(x^1, \ldots, x^n), \qquad u = u(t) \in E^n \\ X &= \operatorname{col}\{x^{(0)}, x^{(1)}, \ldots, x^{(k-1)}\}; \; x^{(i)} = d^i x / dt^i, \qquad 0 \leqslant i \leqslant k-1 \\ L(t, X) &= L(t, x^{(0)1}, \ldots, x^{(0)n}, x^{(1)1}, \ldots, x^{(k-1)n}) \end{aligned} \tag{1.1}$$

$$\begin{aligned} d^l y / dt^l &= H(t, Y) + v, \quad y = \operatorname{col}(y^1, \ldots, y^n), \qquad v = v(t) \in E \\ Y &= \operatorname{col}\{y^{(0)}, \ldots, y^{(l-1)}\}; \; y^{(j)} = d^j y / dt^j, \qquad 0 \leqslant j \leqslant l-1 \\ H(t, Y) &= H(t, y^{(0)1}, \ldots, y^{(0)n}, y^{(1)1}, \ldots, y^{(l-1)n}) \end{aligned} \tag{1.2}$$

Here E^n is an n-dimensional Euclidean space, $u(v)$ is an everywhere finite vector-valued function, measurable for $t \geqslant t_0$, whose scalar square we sum on any interval $[t_1, t_2] \subset [t_0, +\infty]$, called the control of the pursuer (escaper), $X(Y)$ is the phase vector of the pursuer (escaper), $L(t, X)$, $H(t, Y)$ are vector-valued functions continuous together with their first-order partial derivatives in all variables.

We assume that the following condition is satisfied for game (1.1), (1.2): for arbitrary collection $z_* = \{t_*, X_*, Y_*\}$, $t_* \geqslant t_0$, called the (initial) point of the game, and for arbitrary players' controls, the solutions $X(t)$ and $Y(t)$ of Eqs. (1.1) and (1.2), respectively, in the sense of Carathéodory [1], with initial values $X(t_*) = X_*$, $Y(t_*) = Y_*$, exist on the whole interval $[t_*, +\infty]$.

The following constraints are imposed on the players' controls:

$$\int_{t_0}^{+\infty} \rho(t, X(t))(u(t) \cdot u(t))\, dt \leqslant \rho^2 \tag{1.3}$$

$$\int_{t_0}^{+\infty} \sigma(t, Y(t))(v(t) \cdot v(t))\, dt \leqslant \sigma^2 \tag{1.4}$$